2006 Mock AIME 5

Prepared by Jeffrey Wang (paladin8)

1. Suppose n is a positive integer. Let $f(n)$ be the sum of the distinct positive prime divisors of n less than 50 (e.g. $f(12) = 2+3 = 5$ and $f(101) = 0$). Evaluate the remainder when $f(1) + f(2) + \cdots + f(99)$ is divided by 1000.

2. A circle ω_1 of radius $6\sqrt{2}$ is internally tangent to a larger circle ω_2 of radius $12\sqrt{2}$ such that the center of ω_2 lies on ω_1. A diameter AB of ω_2 is drawn tangent to ω_1. A second line l is drawn from B tangent to ω_1. Let the line tangent to ω_2 at A intersect l at C. Find the area of $\triangle ABC$.

3. A *hailstone* number $n = d_1d_2\cdots d_k$, where d_i denotes the ith digit in the base-10 representation of n for $i = 1, 2, \ldots, k$, is a positive integer with distinct nonzero digits such that $d_m < d_{m-1}$ if m is even and $d_m > d_{m-1}$ if m is odd for $m = 1, 2, \ldots, k$ (and $d_0 = 0$). Let a be the number of four-digit *hailstone* numbers and b be the number of three-digit *hailstone* numbers. Find $a + b$.

4. Let m and n be integers such that $1 < m \leq 10$ and $m < n \leq 100$. Given that $x = \log_m n$ and $y = \log_n m$, find the number of ordered pairs (m, n) such that $\lfloor x \rfloor = \lceil y \rceil$. ($\lfloor a \rfloor$ is the greatest integer less than or equal to a and $\lceil a \rceil$ is the least integer greater than or equal to a).

5. Find the largest prime divisor of $25^2 + 72^2$.

6. P_1, P_2, and P_3 are polynomials defined by:

$$P_1(x) = 1 + x + x^3 + x^4 + \cdots + x^{96} + x^{97} + x^{99} + x^{100}$$
$$P_2(x) = 1 - x + x^2 - \cdots - x^{99} + x^{100}$$
$$P_3(x) = 1 + x + x^2 + \cdots + x^{66} + x^{67}.$$

Find the number of distinct complex roots of $P_1 \cdot P_2 \cdot P_3$.

7. A coin of radius 1 is flipped onto an 500×500 square grid divided into 2500 equal squares. Circles are inscribed in n of these 2500 squares. Let P_n be the probability that, given that the coin lands completely within one of the smaller squares, it also lands completely within one of the circles. Let P be the probability that, when flipped onto the grid, the coin lands completely within one of the smaller squares. Let n_0 smallest value of n such that $P_n > P$. Find the value of $\left\lfloor \frac{n_0}{3} \right\rfloor$.

8. Let P be a polyhedron with 37 faces, all of which are equilateral triangles, squares, or regular pentagons with equal side length. Given there is at least one of each type of face and there are twice as many pentagons as triangles, what is the sum of all the possible number of vertices P can have?

9. 13 nondistinguishable residents are moving into 7 distinct houses in Conformistville, with at least one resident per house. In how many ways can the residents be assigned to these houses such that there is at least one house with 4 residents?

10. Find the smallest positive integer n such that $\binom{2n}{n}$ is divisible by all the primes between 10 and 30.

11. Let A be a subset of consecutive elements of $S = \{n, n+1, \ldots, n+999\}$ where n is a positive integer. Define $\mu(A) = \sum_{k \in A} \tau(k)$, where $\tau(k) = 1$ if k has an odd number of divisors and $\tau(k) = 0$ if k has an even number of divisors. For how many $n \leq 1000$ does there exist an A such that $|A| = 620$ and $\mu(A) = 11$? ($|X|$ denotes the cardinality of the set X, or the number of elements in X)

12. Let ABC be a triangle with $AB = 13$, $BC = 14$, and $AC = 15$. Let D be the foot of the altitude from A to BC and E be the point on BC between D and C such that $BD = CE$. Extend AE to meet the circumcircle of ABC at F. If the area of triangle FAC is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

13. Let S be the set of positive integers with only odd digits satisfying the following condition: any $x \in S$ with n digits must be divisible by 5^n. Let A be the sum of the 20 smallest elements of S. Find the remainder upon dividing A by 1000.

14. Let ABC be a triangle such that $AB = 68$, $BC = 100$, and $CA = 112$. Let H be the orthocenter of $\triangle ABC$ (intersection of the altitudes). Let D be the midpoint of BC, E be the midpoint of CA, and F be the midpoint of AB. Points X, Y, and Z are constructed on HD, HE, and HF, respectively, such that D is the midpoint of XH, E is the midpoint of YH, and F is the midpoint of ZH. Find $AX + BY + CZ$.

15. 2006 colored beads are placed on a necklace (circular ring) such that each bead is adjacent to two others. The beads are labeled a_0, a_1, $\ldots$, a_{2005} around the circle in order. Two beads a_i and a_j, where i and j are non-negative integers, satisfy $a_i = a_j$ if and only if the color of a_i is the same as the color of a_j. Given that there exists no non-negative integer $m < 2006$ and positive integer $n < 1003$ such that $a_m = a_{m-n} = a_{m+n}$, where all subscripts are taken $\pmod{2006}$, find the minimum number of different colors of beads on the necklace.