

2006 Mock AIME 1

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1. 2006 points are evenly spaced on a circle. Given one point, find the number of other points that are less than 1 radius distance away from that point.

2. How many 4-digit numbers are there such that the 3rd digit (from the left) is the sum of the first two and the 4th is greater than the sum of the 1st and the 3rd?

3. There are five circles, $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5$ of radius 1, 2, 3, 4, 5, respectively, whose centers lie on the diagonal of a square such that ω_1 is tangent to two sides of the square and ω_2 ; ω_i is tangent to ω_{i-1} and ω_{i+1} for $i = 2, 3, 4$; and ω_5 is tangent to ω_4 and the other two sides of the square. Given that the area of the square is $a + b\sqrt{c}$, where a , b , and c are positive integers and c is squarefree, find $a + b + c$.

4. There are 2006 people in a room. They decide to stack themselves into a pyramid with exactly $2n - 1$ people on the n th row from the top (i.e. the top row has 1 person, the second from the top 3, etc.). There are some number of people left over after this process (all rows must be full to exist and they use the maximum number of people without exceeding the number present). They make another pyramid of the same type. Again, people are left over. If this process is repeated until everyone is in a pyramid, how many people are touching the ground (on the bottom row of a pyramid)?

5. Find the number of positive integers x less than 2500 such that both $\lfloor \log_4 x \rfloor$ and $\lfloor \log_3 x \rfloor$ are prime.

6. Let $d(n)$ denote the sum of the digits of a positive integer n in base ten. Evaluate the remainder when $\sum_{k=10}^{2006} d(k)$ is divided by 1000.

7. Let $f(n)$ denote the number of divisors of a positive integer n . Evaluate $f(f(2006^{6002}))$.

8. Let k be a positive integer and $\{a_n\}$ be a sequence defined as follows:

$$\begin{aligned} a_0 &= k \\ a_{n+1} &= 3a_n + 1 \text{ if } a_n \equiv 1 \pmod{2} \\ a_{n+1} &= \frac{a_n}{2} \text{ if } a_n \equiv 0 \pmod{2}. \end{aligned}$$

Given that $a_5 = 2006$, find the remainder when the sum of all possible values of k is divided by 1000.

9. Let $f(x) = \frac{2}{x^2 - 1}$. Find the largest positive integer n such that

$$f(2) + f(3) + \cdots + f(n) \geq \frac{2006}{1337} - \frac{101}{n} + \frac{99}{n+1}.$$

10. Let A be the area of the convex quadrilateral with vertices on the foci of the ellipses defined by $16x^2 - 320x + 25y^2 - 1500y - 15900 = 0$ and $25x^2 - 1500x + 16y^2 - 320y - 15900 = 0$. Find $\frac{A}{2}$.

11. Let V_1 be the volume of a sphere circumscribed around a regular tetrahedron and v_1 be the volume of a sphere inscribed in the same tetrahedron. Let V_2 be the volume of a sphere circumscribed around a cube and v_2 be the volume of a sphere inscribed in the same cube. Find $\left(\frac{V_1}{v_1}\right)^2 - \left(\frac{V_2}{v_2}\right)^2$.

12. Let m_1 and m_2 be the slopes of the lines through the origin that are tangent to the circle $(x - 2)^2 + (y - 2006)^2 = 1$. Find $\left\lfloor \frac{m_1 + m_2}{10} \right\rfloor$.

13. Al and Bob are playing a game with a coin. They take turns flipping. Each turn, if the player flips a head he flips again. Otherwise, it is the other player's turn. Al goes first. The probability that Al has at least as many heads as Bob after 10 total flips is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $n - m$.

14. Let P be a monic polynomial of degree $n \geq 1$ such that

$$[P(x)]^3 = [P(x)]^2 - P(x) + 6$$

for $x = 1, 2, \dots, n$. Let n_0 be the smallest n such that $P(0) > [P(1) + P(2) + \dots + P(n)]^3$. Find the remainder when $P(0)$ is divided by 1000 given $n = n_0$.

15. Let $ABCD$ be a rectangle and $AB = 24$. Let E be a point on BC (between B and C) such that $DE = 25$ and $\tan \angle BDE = 3$. Let F be the foot of the perpendicular from A to BD . Extend AF to intersect DC at G . Extend DE to intersect AG at H . Let I be the foot of the perpendicular from H to DG . The length of IG is $\frac{m}{n}$, where m and n are relatively prime positive integers. Suppose α is the sum of the distinct prime factors of m and β is the sum of the distinct prime factors of n . Find $\alpha + \beta$.