2007 Bellevue BATH Competition

Party Exam

Scoring: Correct = +4, Blank = −1, Incorrect = −2

Rules: You have 20 minutes for this exam. All answers must be simplified as much as possible or they will be considered incorrect. Only answers on the official answer sheet will be graded.

1. Superman can build a house in an hour. I can build one in two hours. Spiderman does not need a house; he only needs a web. How many minutes would it take for me and Superman to build a neighborhood with 5 houses?

2. Sometimes, knowing how to count is important. Other times, it is not. On this question, however, you must know how to count. Determine the largest prime divisor of the number of words in this problem.

3. Don't you always wonder why there are problems like this? Simplify $\frac{2! \cdot 4! \cdot 6! \cdot 8!}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3! \cdot 3! \cdot 5!}$.

4. You are hosting a pie party and you have invited 100 guests, but you only have a single circular pie and a knife that can cut in straight lines. At least how many cuts will you need to make so that every guest gets a slice of any size?

5. You are spinning in circles when you notice, through eyes in the back of your head, your friend spinning next to you facing in the exact opposite direction you are facing. If you spin once every 5 seconds, and your friend spins once every 7 seconds, how many times do you both face each other in the next hour?

6. Let n be the number of the problem that has answer 6. You are on a number line that goes from 1 to n. You start at 1. If you are on some number k, you may move to either $3k$ or to $k + 1$. In how many different ways can you get to n?

7. Determine the smallest positive integer that has at least 19 positive divisors.

8. An ant is walking along the edges of a tetrahedron, as usual. Today, however, it is feeling quite ambitious. It wants to walk along each edge of the tetrahedron while never retracing its steps. In how many ways can it do so?

9. Three identical little pigs are trapped in a room with walls that are closing in. They notice that there are several ways to escape: the window, the chimney, a trapdoor, the attic, an underground railroad, a submarine, and last but not least the door. Each pig may also choose to escape or to stay and die, but no two pigs can choose the same escape route. In how many ways can the three pigs each escape or die?

10. Find the additive inverse of the sum of the answers to all problems on this test excluding this one and problem 20.

11. Determine the difference between the smallest and largest elements of the following list:

$$\{22, 20 + \sqrt{20 + \sqrt{20 + \sqrt{\cdots}}}, 12\log 4 + 16\log 5 - \log 2048 + 3\log 200, \lfloor 7\pi \rfloor\}.$$

12. The centers of two congruent circles lie along a single diagonal of a square such that each is tangent to the other and two sides of the square. If the square has side length $10 + 10\sqrt{2}$, to the nearest integer what is sum of the areas of the two circles ?

13. The probability that today is rainy is 1/3. Being the gambler I am, I bet that it will be rainy. If it is, I win \$9; otherwise, I lose \$9. I have some inside information, however, so I know that the probability of rain is actually 2/3. By how many dollars did my expected gain increase by having inside information?

14. Find the sum of the prime divisors of $9^5 + 9^4 + 1$.

15. Evaluate $\prod_{m=1}^{12} \left(1 - \cos\left(\frac{\pi m}{6}\right) + i - i\sin\left(\frac{\pi m}{6}\right)\right)$. Yuck!

16. Let $f(x) = 2f(\log x)$ for all $x > 0$ and $f(x) = 1$ for all $x < 0$. Lastly, let $f(0) = -1337$. Find the value of the sum $f(1) + f(2) + \cdots + f(999)$.

17. A cow jumped over the moon. Unfortunately, it was on Jupiter so there were about 60 moons that it could have jumped over, numbered from 1 to 60. Being the detective you are, you want to figure out which moon the cow jumped over. It told you that:

(1) the sum of the digits of the number of the moon is either 8 or 9;
(2) the number of the moon can be written as the sum of two perfect squares;
(3) the number of the moon is the product of two distinct primes.

What is the number of the moon that the cow jumped over?

18. Tile the plane with equilateral triangles of side length 1 (this may take a while, but keep at it). If a circle of radius $\frac{1}{\sqrt{r}}$ is dropped in the plane, it has a $\frac{1}{9}$ probability of not intersecting any equilateral triangle. Find r.

19. 3rd degree Diophantine equations are not easy to solve, but no matter. After all, you have four people. For how many values of $k \leq 1000$ do there exist positive integers a, b, c such that

$$a^2b + a^2c + b^2c + b^2a + c^2a + c^2b - 6abc = k?$$

20. Find the sum of the answers to all problems on this test excluding this one.