2007 Bellevue BATH Competition

Final Boss Battle

Scoring: Points are given after any successful submission of a problem provided that the party has not made 5 previous incorrect submissions of the same problem. 10 points are awarded if it is the party's first submission of the problem, and 2 points deducted for each previous incorrect submission. The minimum number of points awarded for solving a problem is 2.

Rules: You have until the end of the round for this test. You may work with your BATHmates only. All answers must be simplified as much as possible or they will be considered incorrect. Answers must be written down on an official answer sheet in the row and column corresponding to the problem and submission number and then brought by a single BATHmate to the grader, who will provide immediate feedback on the correctness of exactly one of your answers. You must return to your party after receiving feedback and complete the trip again to submit another (or the same) problem. Beware that, if you are successfully attacked by the final boss as you are submitting your problem you must return to your party immediately. Then you may attempt to submit once more.

1. A box has dimensions a, b, and c, which are positive integers satisfying the following conditions: (1) $a < b < c < 100$; and (2) the sum of the digits of a, b, and c is 25. Find the triplet (a, b, c) that maximizes the volume of the box.

2. Define the functions $f(x) = 1 + \sin x + \sin^2 x + \sin^3 x + \cdots$ and $g(x) = f(x) \cdot f(-x)$, where x is in degrees. If $a = \tan(1°)$, find $g(89°)$ in terms of a.

3. You are numbering the pages of your latest novel, but you decide that the digit 0 is pretty useless so you simply ignore any numbers that contain the digit 0. Furthermore, you skip numbers that have three or more of the same digit. If the last page of your novel is numbered 1337, how many pages does it really have?

4. Consider all positive integers of the form $2^a \cdot 5^b$ where a and b are nonnegative integers not exceeding 2007 and let S be the sum of all such numbers. Find the positive integer k that satisfies $10^k \le \frac{S}{2007} < 10^{k+1}$.

5. You are given a deck with 8 cards in it, numbered 1 through 8, and you want to sort it by removing the cards in order. A shuffle operation places the top card of the deck on the bottom, while the remove operation allows you to take out the top or bottom card and place it in the sorted pile. Given that you must remove the cards in increasing order, what is one ordering of the cards that takes the most operations to sort? Give your answer as a sequence of 8 numbers starting from the top of the deck and ending at the bottom.

6. Let A and B be sets of complex numbers such that for any $a \in A$ and $b \in B$ we have $|a + b| = 1$ or $|a - b| = 1$. What is the largest possible value of $\min\{|A|, |B|\}$?

7. A rhombus $ABCD$ is placed along the x-axis so that A is at the origin, B is at $(4,0)$, and $\angle ABC = 60°$. It is rolled along the x-axis three times in the positive direction (i.e. rotated clockwise about the vertex on the x-axis that has a greater x-coordinate until another side of the rhombus is on the x-axis). To the nearest ten, what is the area of the region that the rhombus occupied at some point?

8. Consider an infinite sequence of nonzero real numbers $a_0, a_1, a_2, \ldots$ such that $a_{k+19} = (a_k a_{k+1} \cdots a_{k+18})^{\frac{1}{19}}$ for $k = 0, 1, 2, \ldots$. If the sequence converges to $(a_0^{p_0} a_1^{p_1} a_2^{p_2} \cdots a_{18}^{p_{18}})^{\frac{1}{19}}$, find $p_0^3 + p_1^3 + \cdots + p_{18}^3$.

9. A country has 16 cities numbered from 1 to 16. A traveling salesman at city 1 wants to visit the cities in the country. If he is at city k, however, he may only travel to a city m where either m divides k or k divides m. It takes him a single hour to travel from any city to the next. He wants to find two numbers p and q, where p is the maximum number of cities he can visit, including city 1, without visiting any city twice, and q is the minimum time (in hours) it takes him to visit all of the cities in the country and return to city 1 if he can visit cities multiple times. What is the value of the product pq?

10. You have a set of 2007 unit cubes, which you want to build a structure with. You can connect two cubes by attaching a face of the first cube to a face of the second cube so that they correspond exactly; each cube may be attached to a maximum of six other cubes. A structure must consist of all 2007 unit cubes and cannot have two disjoint parts, i.e. the structure is completely connected. How many possible distinct surface areas are there for the structure?