2007 Mock AIME 6

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1. Let T be the sum of all positive integers of the form $2^r \cdot 3^s$, where r and s are nonnegative integers that do not exceed 4. Find the remainder when T is divided by 1000.

2. Draw in the diagonals of a regular octagon. What is the sum of all distinct angle measures, in degrees, formed by the intersections of the diagonals in the interior of the octagon?

3. Alvin, Simon, and Theodore are running around a 1000-meter circular track starting at different positions. Alvin is running in the opposite direction of Simon and Theodore. He is also the fastest, running twice as fast as Simon and three times as fast as Theodore. If Alvin meets Simon for the first time after running 312 meters, and Simon meets Theodore for the first time after running 2526 meters, how far apart along the track (shorter distance) did Alvin and Theodore start?

4. Let R be a set of 13 points in the plane, no three of which lie on the same line. At most how many ordered triples of points (A, B, C) in R exist such that $\angle ABC$ is obtuse?

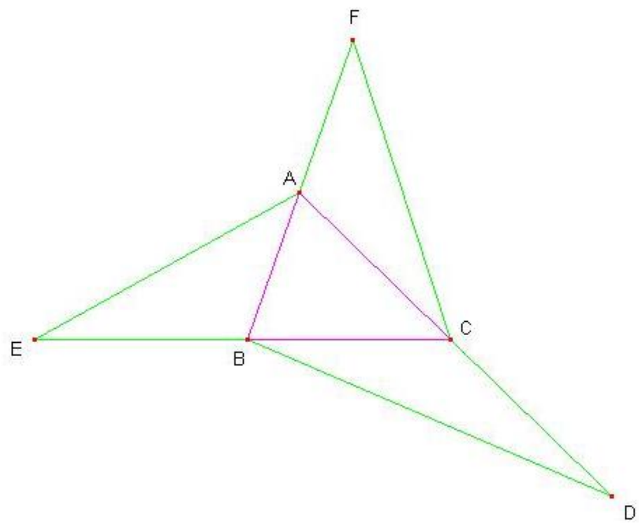
5. Let $S(n)$ be the sum of the squares of the digits of n. How many positive integers $n > 2007$ satisfy the inequality $n - S(n) \leq 2007$?

6. C_1 is a circle with radius 164 and C_2 is a circle internally tangent to C_1 that passes through the center of C_1. $\overline{AB}$ is a chord in C_1 of length 320 tangent to C_2 at D where $AD > BD$. Given that $BD = a - b\sqrt{c}$ where a, b, c are positive integers and c is not divisible by the square of any prime, what is $a + b + c$?

7. Let $P_n(x) = 1 + x + x^2 + \cdots + x^n$ and $Q_n(x) = P_1 \cdot P_2 \cdots P_n$ for all integers $n \geq 1$. How many more distinct complex roots does Q_{1004} have than Q_{1003}?

8. An sequence of positive reals defined by $a_0 = x$, $a_1 = y$, and $a_n \cdot a_{n+2} = a_{n+1}$ for all integers $n \geq 0$. Given that $a_{2007} + a_{2008} = 3$ and $a_{2007} \cdot a_{2008} = \frac{1}{3}$, find $x^3 + y^3$.

9. ABC is a triangle with integer side lengths. Extend $\overline{AC}$ beyond C to point D such that $CD = 120$. Similarly, extend $\overline{CB}$ beyond B to point E such that $BE = 112$ and $\overline{BA}$ beyond A to point F such that $AF = 104$. If triangles CBD, BAE, and ACF all have the same area, what is the minimum possible area of triangle ABC?



10. Given a point P in the coordinate plane, let $T_k(P)$ be the $90°$ counterclockwise rotation of P around the point $(2000 - k, k)$. Let P_0 be the point $(2007, 0)$ and $P_{n+1} = T_n(P_n)$ for all integers $n \geq 0$. If P_m has a y-coordinate of 433, what is m?

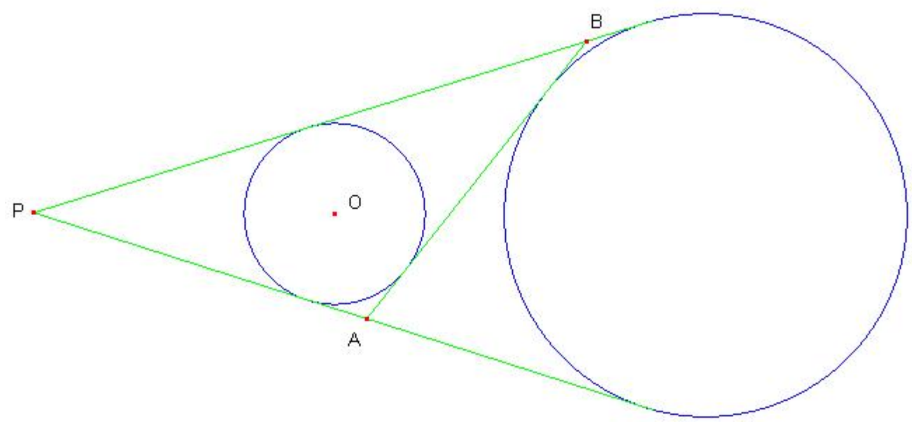
11. Each face of an octahedron is randomly colored blue or red. A caterpillar is on a vertex of the octahedron and wants to get to the opposite vertex by traversing the edges. The probability that it can do so without traveling along an edge that is shared by two faces of the same color is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

12. Let x_k be the largest positive rational solution x to the equation $(2007 - x)(x + 2007^{-k})^k = 1$ for all integers $k \geq 2$. For each k, let $x_k = \frac{a_k}{b_k}$, where a_k and b_k are relatively prime positive integers. If

$$S = \sum_{k=2}^{2007} (2007b_k - a_k),$$

what is the remainder when S is divided by 1000?

13. Consider two circles of different sizes that do not intersect. The smaller circle has center O. Label the intersection of their common external tangents P. A common internal tangent intersects the common external tangents at points A and B. Given that the radius of the larger circle is 11, $PO = 3$, and $AB = 20\sqrt{2}$, what is the square of the area of triangle PBA?



14. A rational $\frac{1}{k}$, where k is a positive integer, is said to be *n-unsound* if its base n representation terminates. Let S_n be the set of all *n-unsound* rationals. The sum of all the elements in the union set $S_2 \cup S_3 \cup \cdots \cup S_{14}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

15. For any finite sequence of positive integers $A = (a_1, a_2, \ldots, a_n)$, let $f(A)$ be be sequence of the differences between consecutive terms of A, i.e. $f(A) = (a_2 - a_1, a_3 - a_2, \ldots, a_n - a_{n-1})$. Let $f^k(A)$ denote f applied k times to A. If all of the sequences A, $f(A)$, $f^2(A)$, $\ldots$, $f^{n-2}(A)$ are strictly increasing and the only term of $f^{n-1}(A)$ is 1, we call the sequence A *superpositive*. How many sequences A with at least two terms and no terms exceeding 18 are *superpositive*?